Zoom Class dated 9/4/2019
Mathematics Blog.
* Content written in { } are explanation not part of defn or Proof.

Exact Differentiation

If an expression is of the type $Mdx + Ndy$ where M & N are functions of x and y or constant then it may be reduced to 'du', where 'u' is a function of x and y, then "$Mdx + Ndy$" is said to be an exact differential

{i.e., $u = f(x, y)$ or constant
~~so~~ $M = g(x, y)$
$N = h(x, y)$

$\Rightarrow Mdx + Ndu = du$ }

Theorem (Statement)

The necessary & sufficient condition that the expression $Mdx + Ndy$ be exact differential is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Proof: {Since the statement has two parts necessary & sufficient we have to prove both parts.

In Necessary we suppose $Mdx + Ndy = du$ to be true and prove $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

In Sufficient we suppose $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ to be true and prove $Mdx + Ndy = du$}

Necessary Condition

Let us suppose that $Mdx + Ndy$ is an exact differential and equal to du.

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$$\Rightarrow \quad Mdx + Ndy = du \quad \text{---(i)}$$

Also we know that

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \quad \text{---(ii)}$$

Comparing (i) & (ii) we get

$$M = \frac{\partial u}{\partial x} \quad \& \quad N = \frac{\partial u}{\partial y}$$

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Now diffrentiating M w.r.t. ~~x~~ we get

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x} \quad \text{---(iii)}$$

Again Diffrentiating 'N' w.r.t. 'x' we get

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \partial y} \longrightarrow (iv)$$

We know that $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$

therefrore from (iii) & (iv) we get

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{Proved}$$

Sufficient Condition

Let us suppose that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

then we have to prove that $M dx + N dy = du$

Let $\int M dx = P \Rightarrow M = \frac{\partial P}{\partial x}$

Now we have $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$\Rightarrow \frac{\partial N}{\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial x}\right) \longrightarrow \{\text{substituting above value of } M\}$$

$$\Rightarrow \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y}\right) \longrightarrow (v)$$

Now integrating both sides of Eqn (v) w.r.t. 'x' we get

$$N = \frac{\partial P}{\partial y} + \phi(y) \quad \text{(vi)}$$ where $\phi(y)$ is a function of y.

{Here instead of Constant we take $\phi(y)$ on integration because N is a function of both x & y and if N is diffrentiated w.r.t. x $\phi(y)$ vanishes}

Now $$M\,dx + N\,dy = \frac{\partial P}{\partial x}dx + \left\{\frac{\partial P}{\partial y} + \phi(y)\right\}dy$$

$$= \frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy + \phi(y)dy$$

$$= d\{P + \psi(y)\}$$ — where $d(\psi(y)) = \phi(y)$

Explaination

{Here we use $\psi(y)$ which is independent of 'x' but dependent on y, but on differentiation it is reduced to $\phi(y)$

$$\Rightarrow d(P) = \frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy$$

$$d(\psi(y)) = 0 \cdot dx + \left\{\frac{\partial(\psi(y))}{\partial y}\right\}dy$$

$$= \phi(y)\,dy$$ }

Now let $P + \varphi(y) = U$

$\Rightarrow M\,dx + N\,dy = du$

<u>Proved</u>

<u>Ques</u> P.T. $(x^3 + 3xy^2)\,dx + (y^3 + 3x^2y)\,dy$ is an exact differential.

<u>Soln</u> Using the ~~above theorem we get the given eqn is in for~~
Given equation is in form of $M\,dx + N\,dy$ so using above theorem we have

$$M = x^3 + 3xy^2 \Rightarrow \frac{\partial M}{\partial y} = 0 + 6yx = 6xy$$

$$N = y^3 + 3x^2y$$

$$\Rightarrow \frac{\partial N}{\partial x} = 0 + 6xy = 6xy.$$

$$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

There fore $(x^3 + 3xy^2)\,dx + (y^3 + 3x^2y)\,dy$ is an exact differential.